Mathematical Proofs
A Transition to Advanced Mathematics

Chapter 0
Communicating Mathematics

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Learning Mathematics

Much of the mathematics that you will encounter in the future is based on what you are about to learn here.

The better you learn the material and the mathematical thought process now, the more you will understand later.

To be sure, any area of study is considerably more enjoyable when you understand it. But getting to that point will require effort on your part.

Learning Mathematics

Mathematics can be learned like any other subject.

What is required is determination and effort.

Creativity is a major part of mathematics.

Discovering mathematics not only contributes to your understanding of the subject but has the potential to contribute to mathematics itself.

Learning Mathematics

Creativity can come in all forms.

The following quote is due to the well-known writer J. K. Rowling (author of the *Harry Potter* novels).

> *Sometimes ideas just come to me. Other times I have to sweat and almost bleed to make ideas come. It's a mysterious process, but I hope I never find out exactly how it works.*

In mathematics our goal is to seek the truth. Finding answers to mathematical questions is important, but we cannot be satisfied with this alone. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

Learning Mathematics

There is joint responsibility here.

As writers, it is our responsibility to give an accurate, clear argument with enough details provided to allow the reader to understand what we have written and to be convinced.

It is the reader's responsibility to know the basics of logic and to study the concepts involved so that a well-presented argument will be understood.

Consequently, in mathematics writing is important, *very* important.

Is it *really* important to write mathematics well?

After all, isn't mathematics mainly equations and symbols? Not at all.

Learning Mathematics

It is not only important to write mathematics well, it is important to write well. You will be writing the rest of your life, at least reports, letters and email. Many people who never meet you will know you only by what you write and how you write.

Mathematics is a sufficiently complicated subject that we don't need vague, hazy and boring writing to add to it.

A teacher has a very positive impression of a student who hands in well-written and well-organized assignments and examinations.

Learning Mathematics

You want people to enjoy reading what you've written. It is important to have a good reputation as a writer. It's part of being an educated person. Especially with the large number of email letters that so many of us write, it has become commonplace for writing to be more casual. Although all people would probably subscribe to this (since it is more efficient), we should know how to write well, formally and professionally, when the situation requires it.

You might think that considering how long you've been writing and that you're set in your ways, it will be very difficult to improve your writing. Not really. If you want to improve, you can and will. Even if you are a good writer, your writing can always be improved. Ordinarily, people don't think much about their writing. Often just thinking about your writing is the first step to writing better.

What Some Well-Known People Have Said About Writing

You don't write because you want to say something; you write because you've got something to say.

F. Scott Fitzgerald, author (*The Great Gatsby*)

Writing comes more easily if you have something to say.

Scholem Asch, author

Either write something worth reading or do something worth writing.

Benjamin Franklin, statesman, writer, inventor

What is written without effort is in general read without pleasure.

Samuel Johnson, writer

What Some Well-Known People Have Said About Writing

In a very real sense, the writer writes in order to teach himself, to understand himself, to satisfy himself; the publishing of his ideas, though it brings gratification, is a curious anticlimax.

Alfred Kazin, literary critic

The skill of writing is to create a context in which other people can think.

Edwin Schlossberg, exhibit designer

Mathematical Writing

We are about to give you some advice, some "pointers," about writing mathematics. Such advice is necessarily subjective. Not everyone subscribes to these suggestions on writing. Indeed, writing "experts" don't agree on all issues. For the present, your instructor will be your best guide. But writing does not follow a list of rules.

Since a number of these writing tips may not make sense (since, after all, we don't even have anything to write yet), it will probably be most useful to return to this chapter periodically as you proceed through what follows.

Using Symbols

Never start a sentence with a symbol.

Writing mathematics follows the same practice as writing all sentences, namely that the first word should be capitalized. This is confusing if the sentence were to begin with a symbol since the sentence appears to be incomplete. Also, in general, a sentence sounds better if it starts with a word.

Instead of writing:

$$x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

write:

$$\text{The equation } x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

Using Symbols

Separate symbols not in a list by words if possible.

Separating symbols by words makes the sentence easier to read and therefore easier to understand.

The sentence:

> With the exception of a, b is the only root of $(x - a)(x - b) = 0$.

would be clearer if it were written as:

> With the exception of a, the number b is the only root of $(x - a)(x - b) = 0$.

Using Symbols

Write out integers as words when they are used as adjectives and when the numbers are relatively small or are easy to describe in words. Write out numbers numerically when they specify the value of something.

There are exactly two groups of order 4.

Fifty million Frenchmen can't be wrong.

There are one million positive integers less than 1,000,001.

Using Symbols

Don't mix words and symbols improperly.

Avoid writing:

Every integer ≥ 2 is a prime or is composite.

It is preferable to write:

Every integer exceeding 1 is prime or composite.

or

If $n \geq 2$ is an integer, then n is prime or composite.

Using Symbols

Although

Since $(x-2)(x-3)=0$, it follows that $x=2$ or 3.

sounds correct, it is not written correctly. It should be:

Since $(x-2)(x-3)=0$, it follows that $x=2$ or $x=3$.

Using Symbols

Avoid using a symbol in the statement of a theorem when it's not needed.

Don't write:

Theorem *Every bijective function f has an inverse.*

Delete "f." It serves no useful purpose. The theorem does not depend on what the function is called. A symbol should not be used in the statement of a theorem (or in its proof) exactly once. If it is useful to have a name for an arbitrary bijective function in the proof (as it probably will be), then "f" can be introduced there.

Using Symbols

Explain the meaning of every symbol that you introduce.

Although what you intended may seem clear, don't assume this. For example, if you write $n = 2k + 1$ and k has never appeared before, then say that k is an integer (if indeed k *is* an integer).

Writing Mathematical Expressions

There will be numerous occasions when you will want to write mathematical expressions in your assignment, such as algebraic equations, inequalities and formulas. If these expressions are relatively short, then they should probably be written within the text of the proof or discussion. (We'll explain this in a moment.) If the expressions are rather lengthy, then it is probably preferred for these expressions to be written as "displays."

Writing Mathematical Expressions

Suppose that we are discussing the Binomial Theorem. (It's not important if you don't recall what this theorem is.) It's possible that what we are writing includes the following passage:

Poor Example

For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

It would probably be better to write the expansion of $(a+b)^4$ as a **display**, where the mathematical expression is placed on a line or lines by itself and is centered. This is illustrated below.

Good Example

If we expand $(a+b)^4$, then we obtain

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Writing Mathematical Expressions

If there are several mathematical expressions that are linked by equal signs and inequality symbols, then we would almost certainly write this as a display. Suppose that we wanted to write $n^3 + 3n^2 - n + 4$ in terms of k, where $n = 2k + 1$. A possible display is given next:

Example

Since $n = 2k + 1$, it follows that

$$\begin{aligned} n^3 + 3n^2 - n + 4 &= (2k+1)^3 + 3(2k+1)^2 - (2k+1) + 4 \\ &= (8k^3 + 12k^2 + 6k + 1) + \\ &\quad 3(4k^2 + 4k + 1) - 2k - 1 + 4 \\ &= 8k^3 + 24k^2 + 16k + 7 \\ &= 2(4k^3 + 12k^2 + 8k + 3) + 1. \end{aligned}$$

Notice how the equal signs are lined up.

Writing Mathematical Expressions

Let's return to the expression

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

for a moment. If we were to write this expression in the text of a paragraph (as we are doing) and if we find it necessary to write portions of this expression on two separate lines, then this expression should be broken so that the first line ends with an operation or comparative symbol such as $+, -, <, \geq$ or $=$. In other words, the second line should *not* begin with one of these symbols. The reason for doing this is that ending the line with one of these symbols alerts the reader that more will follow; otherwise, the reader might conclude (incorrectly) that the portion of the expression appearing on the first line is the entire expression.

Writing Mathematical Expressions

Consequently, write

Good Example

If we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

and not

Poor Example

If we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Writing Mathematical Expressions

If there is an occasion to refer to an expression that has already appeared, then this expression should have been written as a display and labeled as below:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \tag{1}$$

Then we can simply refer to expression (1) rather then writing it out each time.

Common Words and Phrases in Mathematics

There are some words and phrases that appear so often in mathematical writing that it is useful to discuss them.

I *We* *One* *Let's*

I will now show that n is even.

We will now show that n is even.

One now shows that n is even.

Let's now show that n is even.

Common Words and Phrases in Mathematics

These are four ways that we might write a sentence in a proof. Which of these sounds the best to you? It is not considered good practice to use "I" unless you are writing a personal account of something. Otherwise, "I" sounds egotistical and can be annoying. Using "one" is often awkward. Using "we" is standard practice in mathematics. This word also brings the reader into the discussion with the author and gives the impression of a team effort. The word "let's" accomplishes this as well but is much less formal. There is a danger of being *too* casual, however. In general, your writing should be balanced, maintaining a professional style. Of course, there is the possibility of avoiding all of these words:

The integer n is now shown to be even.

Common Words and Phrases in Mathematics

Clearly *Obviously* *Of course* *Certainly*

These and similar words can turn a reader off if what's written is not clear to the reader. It can give the impression that the author is putting the reader down. These words should be used sparingly and with caution. If they *are* used, then at least be certain that what you say is true. There is also the possibility that the writer (a student?) has a lack of understanding of the mathematics or is not being careful and is using these words as a cover-up. This gives us even more reasons to avoid these words.

Common Words and Phrases in Mathematics

Since $\cdots$, *then* $\cdots$

A number of people connect these two words. You should use either "If $\cdots$, then $\cdots$" (should this be the intended meaning) or "Since $\cdots$, it follows that $\cdots$" or, possibly, "Since $\cdots$, we have $\ldots$".

It is correct to write

If n^2 is even, then n is even.

or

Since n^2 is even, it follows that n is even.

or perhaps

Since n^2 is even, n is even.

Avoid writing

Since n^2 is even, then n is even.

In this context, the word "since" can be replaced by "because."

Common Words and Phrases in Mathematics

Therefore *Thus* *Hence* *Consequently* *So*
It follows that *This implies that*

This is tricky. Mathematicians cannot survive without these words. Often within a proof, we proceed from something we've just learned to something else that can be concluded from it. There are many (many!) openings to sentences which attempt to say this. Although each of the words or phrases

Therefore Thus Hence Consequently So

It follows that This implies that

is suitable, it is good to introduce some variety into your writing and not use the same words or phrases any more often than necessary.

Some Closing Comments About Writing

Use good English. Write in complete sentences, ending each sentence with a period (or a question mark when appropriate) and capitalize the first word of each sentence. (Remember: No sentence begins with a symbol!)

Capitalize theorem and lemma as in Theorem 1.15 and Lemma 4.11. (For example, write: In order to verify the truth of Result 3.14, we first prove the following lemma.)